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Approximation Units and Summation of Independent Random Variables

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Let $\{X_n\}$ be a sequence of independent, identically distributed random variables. Assume that X_1 has a density function with $E(X_1)=0$ and $\sigma_n^2(X_1)<\infty$. Call $f_n(x)$ the density of $(1/n)\sum_{i=1}^n X_i$. The Weak Law of Large Numbers gives $\int_{|x|>\varepsilon} f_n(x)\,dx \to 0$ for every $\varepsilon>0$ whereas $\int f_n\,dx=1$. This tells us that $f_n * g \to g$ in the L^1 metric whenever $g\in L^1(-\infty,\infty)$. This can be readily seen in the case when g is continuous with compact support. The general case follows by a density argument as a consequence of Young's inequality. Throughout this paper we show that if in addition the characteristic function of X_1 belongs to some class L^α, large α, then $f_n * g$ converges a.e. to g. Similar results are discussed for the case when $\sigma=\infty$. It is shown that these results can be phrased in terms of a more general theorem concerning approximation units. © 1985 Academic Press, Inc.

1. STATEMENT OF THE MAIN RESULTS

In what follows we will assume that the random variables have a density.

THEOREM 1. *Let $\{X_n\}$ be a sequence of identically distributed and independent random variables such that*

(i) $E(X_1)=0$, $\mathrm{Var}(X_1)=1$.

Call $\phi(u)$ the characteristic function of X_1, that is,

$$\phi(u)=\int_{-\infty}^{\infty} e^{iux} f(x)\,dx;$$

$f(x)$ denotes, as usual, the density of X_1.

Suppose that for some large $\alpha>0$, we have

(ii) $\phi\in L^\alpha(-\infty,\infty)$.

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Denote by $f_n(x)$ the density of $\sum_1^n X_k/n$. Then we have

(j) $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x-y)\, g(y)\, dy = g(x)$ *a.e., whenever* $g \in L^p$, $1 \leqslant p \leqslant \infty$.

(jj) *If* $p > 1$ *and* $g^*(x) = \sup_n |f_n * g|$, *then*

$$\|g^*\|_p < C_p \, \|g\|_p,$$

C_p *independent of* g. *Here,* $\| \ \|_p$ *denotes the* L^p *norm and* $f_n * g$ *denotes the convolution product. Denoting by* $\{x: g(x) > \lambda\}$ *the set of points where* $g > \lambda$, *we have for* L^1 *the result*

(jjj) $|\{x: g^*(x) > \lambda\}| < (C/\lambda)\, \|g\|_1$ *where* $| \ |$ *denotes Lebesgue measure and* C *being a constant independent of* g.

THEOREM 2. *Let* $\{X_n\}$ *be a sequence of independent and identically distributed random variables such that*

(i) $E(X_1) = 0$, $\phi(t) \in L^\alpha(-\infty, \infty)$, *large* α,

(ii) *for some* β, $0 < \beta < 1$ *and* $C > 0$, *we have*

$$\phi'(t) = -C\, |t|^\beta + o(|t|^\beta).$$

Then g^* *satisfies*

(j) $\|g^*\|_p < C_p \, \|g\|_p$, $1 < p \leqslant \infty$.

Here, ϕ, g^*, *and* g *have the same meaning as in Theorem* 1.

THEOREM 3. *Let* $\{k_n(x)\}$ *be a sequence of integrable functions satisfying*

(i) $\int_{-\infty}^{\infty} k_n(x)\, dx = 1$,

(ii) $\int_{-\infty}^{\infty} |k_n(x)|\, dx \leqslant A$.

Suppose that there exists a sequence of functions $h_n(t)$ *such that:*

(iii) $h_n(t)$ *continuous increasing and odd* $(h_n(-t) = -h_n(t))$.

(iv) *If* $m_n(x)$ *denotes the inverse of* $h_n(t)$, *then the* $\{m_n'(x)\}$ *are monotone decreasing if* $x > 0$.

(v) *Suppose that for some* p, $1 < p < \infty$, *we have*

$$\int_{-\infty}^{\infty} |k_n[h_n(t)]\, h_n'(t)|^p (1 + |t|^\gamma)\, dt < A$$

$\gamma > p - 1$ *and* $q = p/(p-1)$.

Then, if $g^* = \sup_n |k_n * g|$, *we have*

(j) $|\{x: g^*(x) > \lambda\}| < (c/\lambda^q) \|g\|_q^q$,

(jj) $\|g^*\|_r < C_r \|g\|_r$, $q < r \leqslant \infty$.

2. Proof of Theorem 1

Consider the kernel function $f_n(x)$ and its homotetic transformation $(1/\sqrt{n}) f_n(x/\sqrt{n})$.

The characteristic function of $(1/\sqrt{n}) f_n(x/\sqrt{n})$ is

$$\phi^n(u/\sqrt{n}). \tag{2.1}$$

Denoting by D the derivative with respect to u, we are going to show that the L^1 norm of the functions

$$D\phi^n(u/\sqrt{n}), \qquad D^1\phi^n(u/\sqrt{n}), \qquad D^2\phi^n(u/\sqrt{n}) \tag{2.2}$$

are bounded from above uniformly.

First, we know that $\phi(u)$ is twice continuously differentiable on account of the existence of the second moment, and this fact automatically implies the same property for $\phi^n(u/\sqrt{n})$.

On the other hand, the existence of the second moment implies

$$\phi(u) = 1 - \tfrac{1}{2}u^2 + o(u^2), \qquad u \to 0 \tag{2.3}$$

(see Ref. [3, p. 485]), and this and the fact that $|\phi(u)|$ has a unique maximum at $u = 0$ assure the validity of the estimates

$$\begin{aligned} &|\phi^n(u/\sqrt{n})| \leqslant e^{-C_0 u^2} && \text{for } |u/\sqrt{n}| \leqslant C_1 \\ &|\phi^n(u/\sqrt{n})| \leqslant (1-\varepsilon)^n && \text{for } |u/\sqrt{n}| \geqslant C_1 \end{aligned} \tag{2.4}$$

where the above bounds hold for all n and for suitable values of the constants C_0, C_1, and ε, $0 < C_0 < \frac{1}{2}$, $0 < \varepsilon < 1$, $C_1 > 0$. From the estimates (2.4) we get

$$\begin{aligned} &\int_{|u| < C_1\sqrt{n}} \left|\phi^n\left(\frac{u}{\sqrt{n}}\right)\right| du \leqslant \int_{-\infty}^{\infty} e^{-C_0 u^2}\, du \\ &\int_{|u| \geqslant C_1\sqrt{n}} \left|\phi^n\left(\frac{u}{\sqrt{n}}\right)\right| du \leqslant (1-\varepsilon)^{n-\alpha} \sqrt{n} \int_{-\infty}^{\infty} |\phi(u)|^\alpha\, du \qquad n > n_0 > \alpha. \end{aligned} \tag{2.5}$$

The first and second derivatives are dealt with in a similar manner. For instance, the second derivative is readily seen to be dominated in the following way:

$$\begin{aligned}\int_{-\infty}^{\infty} \left|D^2\phi^n\left(\frac{u}{\sqrt{n}}\right)\right| du \leqslant C\left(\int_{|u|<C_1\sqrt{n}} |u|^2 e^{-C_0|u|^2}\, du\right) \\ + Cn^{3/2}(E(|X_1|))^2(1-\varepsilon)^{n-\alpha-2}\int_{-\infty}^{\infty} |\phi(u)|^{\alpha}\, du \qquad (2.6) \\ + E(X_1^2)\int_{-\infty}^{\infty} \left|\phi^{n-1}\left(\frac{u}{\sqrt{n}}\right)\right| du.\end{aligned}$$

This holds for large n, say, $n > n_0 > \alpha + 2$. One sees immediately that we have for the last term above the same bound as in (2.5).

From (2.5) and (2.6) and the inversion formula we get

$$(1/\sqrt{n})\, f_n(x/\sqrt{n})(1+x^2) \leqslant M \qquad (2.7)$$

for a constant M independent from x and all n such that $n > n_0$.

The above inequality shows that the kernel function $f_n(x)$ satisfies the estimate

$$f_n(x) \leqslant M\frac{\sqrt{n}}{1+(\sqrt{n}\,x)^2}, \qquad n > n_0. \qquad (2.8)$$

This concludes the proof since the right-hand side is essentially a Poisson kernel. (See [6, Vol. I, pp. 154, 155].) In the reference the result is proved for the case of a bounded interval. The general case follows the same lines.

Proof of Theorem 3. Introducing the change of variables $x = h_n(t)$, $|k_n * g(x_0)|$ is readily seen to be dominated by

$$\begin{aligned}\left(\int_{-1}^{1} |k_n[h_n(t)]\, h_n'(t)|^p\, dt\right)^{1/p} \left(\int_{-1}^{1} |g(x_0-h_n(t))|^q\, dt\right)^{1/q} \\ + \sum_0^{\infty}\left(\int_{2^k<|t|<2^{k+1}} |k_n[h_n(t)]\, h_n'(t)|^p\, dt\right)^{1/p} \qquad (2.9) \\ \times\left(\int_{|t|<2^{k+1}} |g(x_0-h_n(t))|^q\, dt\right)^{1/q}.\end{aligned}$$

The expressions involving g can be dealt with in the following manner:

$$\int_{-2^{k+1}}^{2^{k+1}} |g(x_0-h_n(t))|^q\, dt = \int_{h_n(-2^{k+1})}^{h_n(2^{k+1})} |g(x_0-x)|^q\, m_n'(|x|)\, dx. \qquad (2.10)$$

Call

$$M_q^q(x_0)=\sup_{S>0}\frac{1}{2S}\int_{x_0-S}^{x_0+S}|g(x)|^q\,dx.$$

Using the fact that $m_n'(x)$ is monotone decreasing, an integration by parts gives

$$\begin{aligned}\int_{h_n(-2^{k+1})}^{h_n(2^{k+1})}|g(x_0-x)|^q\,m_n'(|x|)\,dx &\leqslant M_q^q(x_0)\,2\int_0^{h_n(2^{k+1})}m'(x)\,dx\\ &=M_q^q(x_0)\,2^{k+2}.\end{aligned}\tag{2.11}$$

The above step is justified by an application of Lemma 7.1 and Theorem 7.5 of Ref. [6, Vol. I, pp. 154, 155]. Similar estimates hold for

$$\left(\int_{-1}^{1}|g(x_0-h_n(t)|^q\,dt\right)^{1/q}.\tag{2.12}$$

On account of (2.10), (2.11), and (2.12) we get for (2.9) the estimate

$$\begin{aligned}C,M_q(x_0)\Bigg\{&\left(\int_{-1}^{1}|k_n[h_n(t)]\,h_n'(t)|^p\,dt\right)^{1/p}\\ &+\left(\int_{|t|>1}|k_n[h_n(t)]\,h_n'(t)|^p\,t^r\,dt\right)^{1/p}\Bigg\}.\end{aligned}\tag{2.13}$$

Equation (2.13) and an application of the Hardy–Littlewood theorem (see [6, Vol. I, p. 32]) finish the proof.

Proof of Theorem 2. The proof of this theorem will be a consequence of Theorem 3. If $f_n(x)$ denotes the density of $(1/n)\sum_1^n X_i$ we shall show that if n is large enough and $q>2$ we will have

$$\left\|\frac{1}{n^\theta}f_n\left(\frac{x}{n^\theta}\right)(1+|x|)\right\|_q<A_q,\qquad \theta=\frac{\beta}{1+\beta}.\tag{2.14}$$

That is, the role of the $h_n(t)$ will be played by t/n^θ. The above inequality will be proven by using the Haussdorf–Young theorem for Fourier integrals (see [6, Vol. II, p. 254]) after proving an associated inequality for the characteristic functions.

The characteristic associated with the density $(1/n^\theta)f_n(x/n^\theta)$ is

$$\left\{\phi\left(\frac{u}{n^\gamma}\right)\right\}^n,\qquad \gamma=\frac{1}{1+\beta}.\tag{2.15}$$

Condition (ii) of the hypothesis implies that for $n > n_0$

$$\left|\phi\left(\frac{u}{n^{\gamma}}\right)^{n}\right| \leqslant C e^{-C|u|^{1+\beta}} \tag{2.16}$$

if $|u| < \delta n^{\gamma}$ and

$$\left|\phi\left(\frac{u}{n^{\gamma}}\right)^{n}\right| \leqslant (1-\varepsilon)^{n-\alpha}\left|\phi\left(\frac{u}{n^{\gamma}}\right)\right|^{\alpha} \tag{2.17}$$

if $|u| > \delta \cdot n^{r}$. In the same manner we obtain for the derivative $D\phi^{n}(u/n^{\gamma})$ the estimates

$$C\,|u|^{\beta}\, e^{-C|u|^{1+\beta}}, \qquad n > n_0,\ |u| < \delta \cdot n^{\gamma} \tag{2.18}$$

and

$$C n^{\gamma}(1-\varepsilon)^{n-\alpha}\left|\phi\left(\frac{u}{n^{\gamma}}\right)\right|^{\alpha}, \qquad |u| > \delta n^{\gamma}. \tag{2.19}$$

The inequalities (2.16), (2.17), (2.18), and (2.19) give

$$\begin{aligned} \left\|\phi^{n}\left(\frac{u}{n^{\gamma}}\right)\right\|_{p} &< A_1 \\ \left\|D\,\phi^{n}\left(\frac{u}{n^{\gamma}}\right)\right\|_{p} &< A_1 \end{aligned} \tag{2.20}$$

for $n > n_0$, $p = q/(q-1)$. An application of the Haussdorf–Young inequality gives (2.14) from (2.20). This concludes the proof of Theorem 2.

3. Final Remarks

Theorem 3, or a similar type of result, seems to be the tool to handle convolution kernels of the form

$$C_n \int_{-\infty}^{\infty} \varphi^{n}(x-t)\, g(t)\, dt \tag{3.1}$$

where $\varphi(t)$ is such that $\varphi(0) = 1$, $\varphi(t)$ continuous, integrable, and $|\varphi(t)| < 1$ if $|t| > 0$. C_n is a normalizing factor so that

$$C_n \int_{-\infty}^{\infty} \varphi^{n}(t)\, dt = 1. \tag{3.2}$$

In the special case when

$$\varphi(t)=1-|t|^{\beta}+o(|t|^{\beta}) \tag{3.3}$$

the natural choice of the functions $\{h_n(t)\}$ is $h_n(t)=t/n^{1/\beta}$. Operators of the type (3.1) were first treated by Hans Hahn in a famous memoir published by the Academy of Sciences of Vienna in 1916; its translated title is "On the Representation of Functions through Singular Integrals." Operators of a similar type have been treated independently but later by Perron [4] and Widder [5].

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